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FORESTS & GROVES: MINIMAL GAUGE INVARIANT CLASSES OF TREE DIAGRAMS IN GAUGE THEORIES

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We describe the explicit construction of *groves*, the smallest gauge invariant classes of tree Feynman diagrams in gauge theories.

1 Introduction

Calculations of cross section with many-particle final states remain challenging despite all technical advances and it is crucial to be able to concentrate on the important parts of the scattering amplitude for the phenomena under consideration. In gauge theories, however, it is impossible to naively select a few signal diagrams and to ignore the rest. The same subtle cancellations among the diagrams in a gauge invariant subset that lead to the celebrated good high energy behavior of gauge theories come back to haunt us if we accidentally select a subset of diagrams that is not gauge invariant. Results of such a calculation have *no* predictive power, because they depend on unphysical parameters introduced during the gauge fixing of the Lagrangian.

The subsets of Feynman diagrams selected for any calculation must therefore form a *gauge invariant subset*, i. e. they must satisfy the Ward- and Slavnov-Taylor-identities by themselves to ensure the cancellation of contributions from unphysical degrees of freedom. Since not all diagrams in a gauge invariant subset have the same pole structure, a selection based on "signal" or "background" will *not* suffice.

In abelian gauge theories, such as QED, the classification of gauge invariant subsets is straightforward and can be summarized by the requirement of inserting any additional photon into *all* connected charged propagators. This situation is similar for gauge theories with simple gauge groups, the difference being that the gauge bosons are carrying charge themselves. For non-simple gauge groups like the Standard Model (SM), the classification of gauge invariant subsets is more involved.

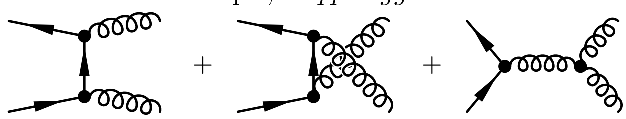
We present the explicit construction of *groves*[1], the smallest gauge invariant classes of tree Feynman diagrams in gauge theories. This construction is applicable to gauge groups with any number of factors, which can even be mixed, as in the SM. Furthermore, it does not require a summation over complete multiplets and can therefore be used in flavor physics when members of weak isospin doublets are detected separately.

In unbroken gauge theories, the permutation symmetry of external gauge quan-

tum numbers can be used to subdivide the scattering amplitude corresponding to a grove further into gauge invariant sub-amplitudes. In this decomposition, each Feynman diagram contributes to more than one sub-amplitude. It is not yet known how to perform a similar decomposition systematically in the SM, because the entanglement of gauge couplings and gauge boson masses complicates the structure of the amplitudes.

2 Gauge Cancellations

Even if the general arguments are well known, the intricate nature of the cancellations of unphysical contributions complicates the development of systematic calculational procedures. Nevertheless, phenomenology needs numerically well behaved matrix elements, which should be as compact as possible and the development of an automated procedure remains a formidable challenge. The selection of gauge invariant subsets of Feynman diagrams described here is one necessary step towards this goal. One particular problem is that the Ward identities relate diagrams with different pole structure. For example, in $q\bar{q} \to gg$



the numerator factors cancel parts of denominators and the Ward identity is satisfied only by the sum and not by individual diagrams.

Therefore, the physical amplitudes are determined by an intricate web of kinematical structure and gauge structure. Nevertheless, the identification of partial sums of Feynman diagrams that are gauge invariant by themselves is of great practical importance. It is more economical to spent the time improving Monte Carlo statistics for the important pieces of the amplitude than to calculate the complete amplitude all the time. This requires the identification of the smallest gauge invariant part of the amplitude that contains the important pieces. Secondly, different parts of the amplitude are characterized by different scales for radiative corrections, but different scales can only be used if the parts correspond to separately gauge invariant pieces.

3 Flavor Selection Rules

One method of identifying gauge invariant subsets is to identify subsets that contribute to a particular final state[2,3]. These subsets are not minimal, but they show that selection rules of flavor symmetries that commute with the gauge group are useful tools. The simplest example is Bhabha scattering: the s- and t-channel diagrams have to be gauge-invariant separately, because both $e^+e^- \to \mu^+\mu^-$ and $e^+\mu^- \to e^+\mu^-$ are physical processes with gauge invariant amplitudes.

A less trivial example are the three separately gauge invariant sets in $ud \to ud$ scattering shown in figure 1. The separate gauge invariance of the gluon exchange diagram is obvious because the strong coupling can be changed without violating gauge invariance. The charged current diagram is separately gauge-invariant, because we may assume that the CKM mixing matrix is diagonal and then the charged

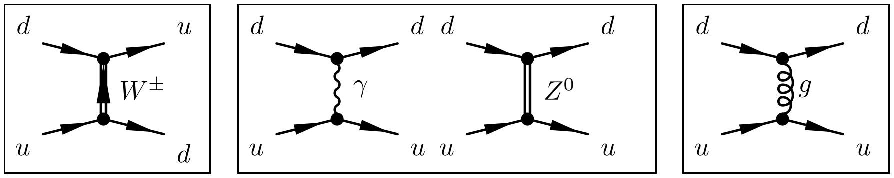


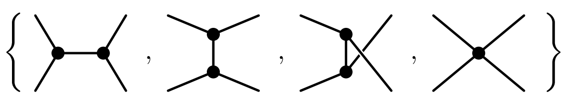
Figure 1. Gauge invariant subsets of Feynman diagrams in $ud \to ud$ scattering.

current diagram is absent in $us \to us$ scattering, which is related to $ud \to ud$ by a horizontal symmetry that commutes with the gauge group.

4 Forests

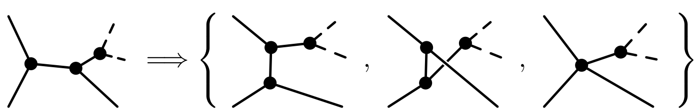
To develop tools for taming the combinatorics, we start from unflavored scalar ϕ^3- and ϕ^4-theory. Since there are no selection rules, the diagrams S_1, S_2, and S_3 in

$$T_4 = \{t_4^{S,1}, t_4^{S,2}, t_4^{S,3}, t_4^{S,4}\} = \tag{1}$$



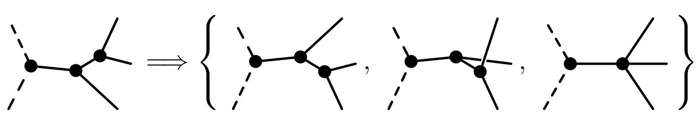
must have the same coupling strength to ensure crossing invariance. If there are additional symmetries, as in the case of gauge theories, the coupling of $t_4^{S,4}$ will also be fixed relative to $t_4^{S,1}$, $t_4^{S,2}$, and $t_4^{S,3}$.

We call each exchange $t \leftrightarrow t'$ of two members of the set T_4 of all tree graphs with four external particles an *elementary flip*. The elementary flips define a trivial relation $t \circ t'$ on T_4, which is true, if and only if t and t' are related by a flip. The relation $\circ$ is trivial on T_4 because *all* pairs are related by an elementary flip. However, the elementary flips in T_4 induce *flips* in T_5 (the set of all tree diagrams with five external particles) if they are applied to an arbitrary four particle subdiagram



(2a)

Obviously, there is more than one element of T_4 embedded in a particular $t \in T_5$ and the same diagram is member of other quartets, e. g.



(2b)

There are 15 five-point tree diagrams in ϕ^3 and only four other diagrams can be reached from any diagram by elementary flips. Thus there is a non-trivial mathematical structure on the set T_5 with the relation $\circ$, visualized by the graph on the left hand side of figure 2.

Thus the trivial relation on T_4 has a non-trivial natural extension to the set T_n of all n-point tree diagrams: $t \circ t'$ is true if and only if t and t' are identical up to a single flip of a four-point subdiagram

$$t \circ t' \Longleftrightarrow \exists t_4 \in T_4, t'_4 \in T_4 : t_4 \circ t'_4 \wedge t \setminus t_4 = t' \setminus t'_4 \tag{3}$$

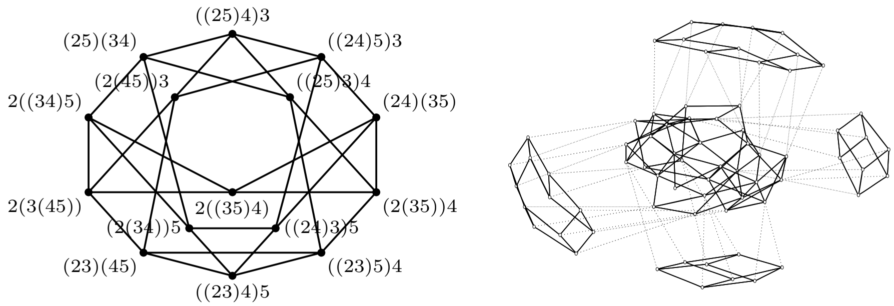


Figure 2. Left: the forest F_5 of the 15 five-point tree diagrams in unflavored ϕ^3-theory. The diagrams are specified by fixing vertex 1 and using parentheses to denote the order in which lines are joined at vertices. Right: the forest of size 71 for the process $\gamma\gamma \to u\bar{d}d\bar{u}$ in the SM (without QCD, CKM mixing and masses in unitarity gauge) with one grove of size 31, two of size 12 and two of size 8. Solid lines represent gauge flips and dotted lines represent flavor flips.

Note that this relation is not transitive and therefore not an equivalence relation. Instead, this relation allows us to view the elements of T_n as the vertices of a graph F_n, where the edges of the graph are formed by the pairs of diagrams related by a single flip

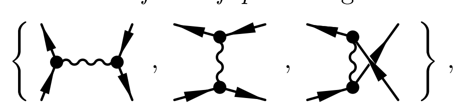
$$F_n = \left\{ (t,t') \in T_n \times T_n \middle| t \circ t' \right\} . \tag{4}$$

Theorem 1 *The unflavored forest F_n is connected for all n.*[1,4]
Already the simplest non-trivial example of an unflavored forest, shown on the left hand side of figure 2, displays an intriguing symmetry structure: there are 120 permutations of the vertices of F_5 that leave this forest invariant.

5 Groves

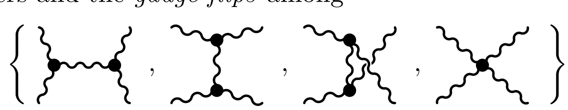
The construction of the groves is based on the observation that the flips in gauge theories fall into two different classes: the *flavor flips* among

$$\{t_4^{F,1}, t_4^{F,2}, t_4^{F,3}\} = \tag{5}$$



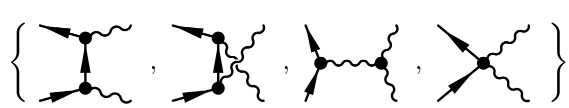
which involve four matter fields that carry gauge charge and possibly additional conserved quantum numbers and the *gauge flips* among

$$\{t_4^{G,1}, t_4^{G,2}, t_4^{G,3}, t_4^{G,4}\} = \tag{6a}$$



or

$$\{t_4^{G,5}, t_4^{G,6}, t_4^{G,7}, t_4^{G,8}\} = \tag{6b}$$



which also involve external gauge bosons (the diagram $t_4^{G,8}$, is only present for scalar matter fields). The flavor flips (5) are special because they can be switched

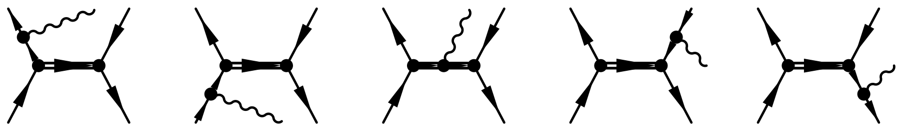


Figure 3. All five diagrams in $u\bar{d} \to c\bar{s}\gamma$ are in the same grove, because they are connected by gauge flips passing through the diagram in the center. In contrast, in $e^+e^- \to \mu^+\mu^-\gamma$, the center diagram is missing and there are two separate groves.

off without spoiling gauge invariance by a horizontal symmetry orthogonal to the gauge group. This suggests to introduce two relations:

$$t \bullet t' \Longleftrightarrow t \text{ and } t' \text{ related by a } \textit{gauge} \text{ flip} \tag{7a}$$

$$t \circ t' \Longleftrightarrow t \text{ and } t' \text{ related by a } \textit{flavor or gauge} \text{ flip.} \tag{7b}$$

These define two different graphs on the same set $T(E)$ of all Feynman diagrams:

$$F(E) = \left\{(t,t') \in T(E) \times T(E) \middle| t \circ t'\right\} \tag{8a}$$

$$G(E) = \left\{(t,t') \in T(E) \times T(E) \middle| t \bullet t'\right\}. \tag{8b}$$

As we have seen, it is in general not possible to connect all pairs of diagrams in the *gauge forest* $G(E)$ by a series of gauge flips and there is more than one connected component (see figure 3). We call the connected components $G_i(E)$ of $G(E)$ the *groves* of E. Since $t \bullet t' \Rightarrow t \circ t'$, we have $\bigcup_i G_i(E) = G(E) \subseteq F(E)$, i. e. the groves are a *partition* of the gauge forest.

Theorem 2 *The forest $F(E)$ for an external state E consisting of gauge and matter fields is connected if the fields in E carry no conserved quantum numbers other than the gauge charges. The groves $G_i(E)$ are the minimal gauge invariant classes of Feynman diagrams.*[1,4]

The forests and groves are very symmetrical and their automorphism groups turn out to be surprisingly large: the automorphism group of the forest $F(\gamma\gamma \to u\bar{d}d\bar{u})$ on the right hand side of figure 2 has 128 elements.

Acknowledgments

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